8/27 Recitation Solutions

1. Let

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ .5 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 9 \\ 7 \\ 3 \\ -2 \end{bmatrix}$$

   Use Julia to perform the following calculations

   (a) Determine the linear combination of $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ with weights $2, -3, \frac{5}{2}$.

   Code:
   a = [-1 ; .5 ; 0 ; 0]
   b = [2 ; 0 ; -1 ; 3]
   c = [9 ; 7 ; 3 ; -2]
   2*a-3*b+5/2*c

   Output: $2\mathbf{v}_1 - 3\mathbf{v}_2 + \frac{5}{2}\mathbf{v}_3 = \begin{bmatrix} 14.5 \\ 18.5 \\ 10.5 \\ -14 \end{bmatrix}$

   (b) Determine the dot product $\mathbf{v}_1 \cdot \mathbf{v}_2$.

   Code:
   using LinearAlgebra
   dot(a,b)

   Output: $\mathbf{v}_1 \cdot \mathbf{v}_2 = -2$

2. Recall the distributive property: $c(\mathbf{v}+\mathbf{w}) = c\mathbf{v}+c\mathbf{w}$ for all vectors $\mathbf{v}, \mathbf{v} \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$.

   (a) Use Julia to verify this property for two symbolic vectors in $\mathbb{R}^4$.

   Code:
   using SymPy
   v = [symbols("v$i") for i=1:4]
   w = [symbols("w$i") for i=1:4]
   c = symbols("c")
   c*(v-w)==c*v-c*w

   Output: true

(b)

(c) Pick two concrete vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$ and a concrete scalar c. Verify the property is true geometrically for these choices; that is, use Julia to plot both $c(\mathbf{v}+\mathbf{w})$ and $c\mathbf{v}+c\mathbf{w}$ and compare.

Code for $c(\mathbf{v}_1+\mathbf{v}_2)$ plot:

```
using Plots
k=2
a=4;
b=1;
c=1;
d=3;
x_vals = [0 0 c a 0 0; a c (a+c) (a+c) k*(a+c) (a+c)]
y_vals = [0 0 d b 0 0; b d (b+d) (b+d) k*(b+d) (b+d)]
plot(x_vals, y_vals, arrow = true, color = [:blue :red :blue :red :green :purple],
legend = :none, xlims = (0, 10), ylims = (0, 10),
annotations = [(a+.3, b, "v"),
(c+.3, d, "w"),
((a+c-1.5), (b+d-2), "v+w"),
((k*a+k*c-1.5), (k*b+k*d-2), "2(v+w)")],
xticks = 0:1:10, yticks = 0:1:10,
framestyle = :origin)
```

Code for $c\mathbf{v}_1+c\mathbf{v}_2$ plot:

```
x_vals = [0 0 k*c k*a 0; k*a k*c k*(a+c) k*(a+c) k*(a+c)]
y_vals = [0 0 k*d k*b 0; k*b k*d k*(b+d) k*(b+d) k*(b+d)]
plot(x_vals, y_vals, arrow = true, color = [:blue :red :blue :red :green],
legend = :none, xlims = (0, 10), ylims = (0, 10),
annotations = [(a+.3, b, "2v"),
(c+.3, d, "2w"),
((k*a+k*c-2), (k*b+k*d-2), "2v+2w")],
xticks = 0:1:10, yticks = 0:1:10,
framestyle = :origin)
```

(d) Prove the distributive property in general, by hand.

Let $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, and $c \in \mathbb{R}$.

Then

$$c(\mathbf{v}+\mathbf{w}) = c\Big(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}\Big) = c\begin{bmatrix} v_1+w_1 \\ \vdots \\ v_n+w_n \end{bmatrix} = \begin{bmatrix} c(v_1+w_1) \\ \vdots \\ c(v_n+w_n) \end{bmatrix}$$
$$= \begin{bmatrix} cv_1+cw_1 \\ \vdots \\ cv_n+cw_n \end{bmatrix} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix} + \begin{bmatrix} cw_1 \\ \vdots \\ cw_n \end{bmatrix} = c\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + c\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = c\mathbf{v}+c\mathbf{w}$$

Looking Ahead

3. Let $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Sketch all multiples of $\mathbf{v}$ in the coordinate plane. The set of all such vectors is called the **span** of $\mathbf{v}$.

   The multiples of $\mathbf{v}$ make up the line containing $\mathbf{v}$ (i.e. the line given by equation $y = -x$).

4. Let $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$. Express $\mathbf{b}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$ in two ways:
   (a) Use Brute-force. Find two scalars c_1 and c_2 such that $c_1\mathbf{v} + c_2\mathbf{w} = \mathbf{b}$. You can do this by hand or use Julia to compute $c_1\mathbf{v} + c_2\mathbf{w}$ for different input scalars.
   (b) Develop a generalizable method. Translate this linear combination equation into a system of two equations that you can solve systematically using algebra.

$$c_1\mathbf{v} + c_2\mathbf{w} = \mathbf{b}$$
$$c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$
$$\begin{bmatrix} c_1+3c_2 \\ -c_1+5c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

   We have the system

$$c_1 + 3c_2 = 2$$
$$-c_1 + 5c_2 = 6$$

   Add these equations yields $8c_2 = 8$ or $c_2 = 1$. Plugging back into the first equation, we find $c_1 = -1$. So the weights in the linear combination equation are $c_1 = -1$ and $c_2 = 1$; hence

$$-\mathbf{v} + \mathbf{w} = \mathbf{b}$$

5. Open the **Matrix** notebook to see the commands you will be using next time. Play around with them.